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Inflation and Inflation Uncertainty in the Democratic Republic of Congo

Gabriel K. Midagu* Ishara Musimwa† Yves B. Togba‡

Abstract

Using monthly inflation data from the Democratic Republic of Congo (DRC), this study investigates the effect of inflation uncertainty on inflation over time using Stochastic Volatility in Mean with Time-Varying Parameters model. We find that the effects of inflation uncertainty have been persistent in DRC. For example, our results show that inflation uncertainty—measured by the stochastic volatility—has a positive impact on inflation across all of the sample starting from 2007 and the underlying impact exhibits temporal evolution. These findings have implications for the conduct of both monetary and fiscal policy in the DRC.

Keywords: Inflation, Uncertainty.

JEL classification: E23, E24, H23, Q54, Q58.

1 Introduction

The relationship between inflation and its uncertainty has remained a focal point of macroeconomic research over several decades, attracting both theoretical reflection and empirical testing. Despite this sustained interest, scholars have yet to converge on a definitive understanding of how these two variables interact. The ambiguity lies

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not only in identifying the causal direction but also in the precise mechanisms and conditions under which one influences the other.

Uncertainty surrounding future inflation is widely acknowledged as one of the primary costs of inflation itself, primarily because it obscures the decision-making environment for economic agents. In technical terms, uncertainty refers to the unpredictable volatility of a variable—a combination of what can be forecasted and what remains inherently erratic (Crawford and Kasumovich, 1996; Grier and Perry, 1998). This characteristic makes inflation uncertainty a significant concern for both policymakers and investors alike.

From a policy standpoint, the inflation-uncertainty nexus is critically important. If inflation uncertainty can influence real economic outcomes, it follows that governments, through monetary policy, hold some sway over broader macroeconomic performance. The effects of uncertainty manifest in two ways. On the *ex ante* side, inflation volatility affects forward-looking decisions such as investment and savings by elevating risk premiums and raising long-term interest rates (Golob et al., 1994; Huybens and Smith, 1999). On the *ex post* side, any divergence between expected and actual inflation introduces redistributive effects, especially in contracts fixed in nominal terms.

Two dominant theoretical paradigms aim to explain this relationship. The Friedman–Ball view (Friedman, 1977; Ball, 1992) suggests that higher inflation exacerbates uncertainty by making future policy paths harder to predict, thereby increasing inflation volatility. In contrast, the models developed by Cukierman and Meltzer (1986), and later by Cukierman (1992), argue that rising uncertainty can itself cause policymakers to pursue higher average inflation. Their reasoning, grounded in the Barro–Gordon framework, posits that uncertainty incentivizes inflation surprises aimed at short-term economic stimulation. Empirical studies by Holland (1995), Grier and Perry (1998), and Berument and Nergiz (2005) lend support to this perspective.

Methodologically, early empirical studies relied on backward-looking metrics such as rolling standard deviations or subjective forecast dispersions. More recent approaches use Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models to estimate the one-step-ahead conditional variance of inflation shocks.

GARCH models are advantageous because they allow for direct testing of time-variation in volatility (Grier and Perry, 1998; Evans, 1991). Nonetheless, these mod-

els are limited by their assumption of constant unconditional variance and their insensitivity to regime changes. To overcome these limitations, stochastic volatility models have gained traction in recent years. These models provide a more flexible structure by allowing for time-varying volatility that can accommodate structural shifts and evolving macroeconomic conditions. Despite their relatively limited empirical use, they offer a promising alternative for capturing the complex dynamics between inflation and its uncertainty.

In this paper, we aim to explore how inflation uncertainty feeds back into actual inflation outcomes. For this, we employ the Stochastic Volatility in Mean model with Time-Varying Parameters, following the approach of [Chan \(2017\)](#). This framework allows for a dynamic investigation of the inflation-uncertainty relationship and provides deeper insight into the mechanisms through which macroeconomic volatility can influence inflationary trends.

What makes this investigation particularly pertinent is our choice of the Democratic Republic of Congo (DRC) as the empirical setting. As a developing country with a history of macroeconomic instability, currency depreciation, and reliance on imported goods, the DRC offers a fertile ground for studying the implications of inflation uncertainty. Developing economies like the DRC are often more exposed to external shocks, have weaker institutional frameworks, and face greater challenges in implementing credible and consistent monetary policies. In such contexts, uncertainty around future inflation not only undermines household purchasing power and investor confidence, but also complicates government efforts in fiscal planning and debt sustainability. Moreover, inflation in the DRC tends to be more volatile than in advanced economies due to its exposure to commodity price fluctuations, exchange rate instability, and limited monetary policy transmission mechanisms. Understanding the feedback loop between inflation and its uncertainty can thus provide critical insights for policymakers aiming to improve macroeconomic management, foster investor confidence, and protect vulnerable populations from the destabilizing effects of price volatility.

Using a Stochastic Volatility in Mean model with Time-Varying Parameters, our analysis of the Democratic Republic of Congo from January 2007 to December 2023 uncovers dynamic patterns in both inflation volatility and its impact on inflation itself. First, the estimated inflation uncertainty reveals a sharp increase in volatility between 2007 and 2009, followed by a temporary decline in 2010 and a sustained period of relatively low inflation volatility between 2012 and 2016, reflecting a phase

of improved macroeconomic stability. However, volatility resurged from late 2016 through 2018, declined briefly in early 2019, then rose significantly between 2019 and 2020, before easing during the Covid-19 crisis. Notably, a renewed surge in volatility emerged post-2021 and persisted until the end of the sample in December 2023, highlighting ongoing economic fragility. In parallel, our analysis of the time-varying effect of inflation uncertainty on inflation, shows a consistent and positive influence of uncertainty on inflation throughout the entire period. This impact underscores the persistent and evolving nature of the relationship: inflation uncertainty not only accompanies inflation fluctuations but actively contributes to their amplification. Since 2010, this relationship has remained robust, emphasizing the importance of accounting for such time-varying dynamics in monetary policy formulation in contexts like the DRC.

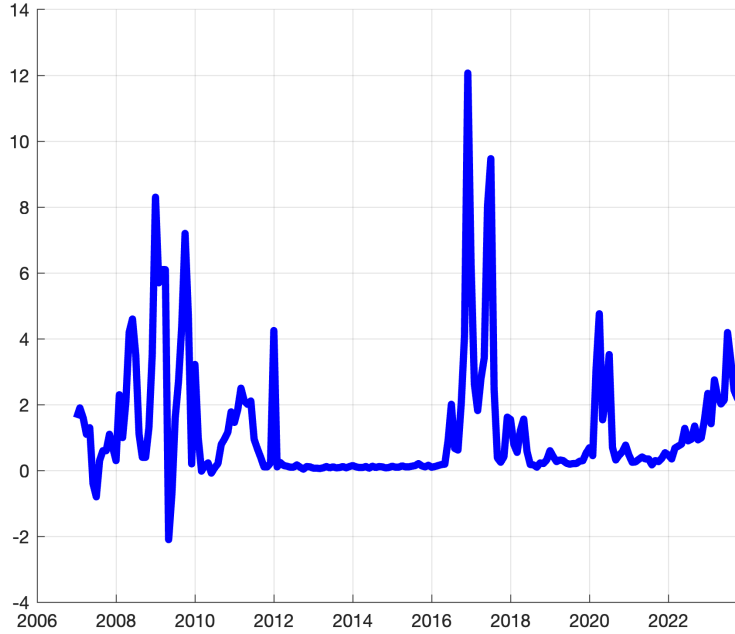
Due to our focus on simple, concrete take-aways for applied practice, this paper contributes to the expanding literature on Inflation and inflation uncertainty (Holland, 1995; Grier and Perry, 1998; Berument and Nergiz, 2005; Berument et al., 2009; Chan, 2017), with a particular focus DR. Congo economy. We differ from previous literature by employing a model with a particular specification: unlike models with constant coefficients in the conditional mean, our time-varying parameter model allows us to assess whether the relationship between inflation and inflation uncertainty has evolved over time in the DRC. In addition, we use data from the Democratic Republic of Congo, a developing country with distinct structural characteristics.

2 Data

The data set used in this paper come from the Central Bank of Congo. The seasonally adjusted monthly data of the Consumer Price Index is used to calculate inflation. The sample period spans 204 months, from January 2007 to December 2023. The Figure (1) plots the dynamics of inflation.

From January 2007 to December 2023, the inflation dynamics in the Democratic Republic of Congo (DRC) have shown notable volatility, reflecting both internal macroeconomic challenges and external shocks. The average monthly inflation rate during this period stood at approximately 1.22%, with significant fluctuations. Periods of high inflation were especially pronounced during early 2009, where monthly rates exceeded 8%, notably in January (8.3%) and sustained above 5% through April, largely driven by the global financial crisis and its repercussions on commodity-exporting economies like the DRC. Conversely, there were several deflationary episodes, particularly from mid-2007 and in 2009, when inflation dropped

Figure 1 – Monthly Inflation



Source : Central Bank of Congo

below zero, indicating contractionary pressures or abrupt corrections in prices—most notably in May 2009 with a record low of -2.1%. These oscillations suggest a fragile macroeconomic framework, sensitive to both global commodity prices and domestic policy shifts. The post-2010 period appears more stable, though short-lived spikes persisted, pointing to the intermittent effectiveness of monetary and fiscal interventions in anchoring inflation expectations. Overall, the inflation profile of the DRC during this timeframe underscores structural vulnerabilities and the need for sustained reforms to ensure price stability.

3 Econometric Framework

We follow the model by [Chan \(2017\)](#) which extend the standard SVM model to allow for time-varying coefficients in the conditional mean as this feature is generally recognized as important for typical macroeconomic time series. We use the underlying model to investigate if inflation uncertainty affects the level of the DRC inflation.

The specificity of this model is that in contrast to models with constant coefficients in the conditional mean time-varying, the parameter model allows us to assess if the inflation-inflation uncertainty relationship has changed over time in DRC.

The time-varying parameter stochastic volatility in mean (TVP-SVM) model explained in this section generalizes the original specification in [Koopman and Hol Uspensky \(2002\)](#).

Thus, we consider a time-varying parameter model with stochastic volatility where the stochastic volatility also enters the conditional mean equation. Specifically, let consider π_t , our time series of interest, such that

$$\pi_t = \mathbf{x}_t' \beta_t + \alpha_t e^{h_t} + \varepsilon_t^\pi \quad \varepsilon_t^\pi \sim \mathcal{N}(0, e^{h_t}) \quad (1)$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \varepsilon_t^h \quad \varepsilon_t^h \sim \mathcal{N}(0, \sigma^2) \quad (2)$$

where $\mathbf{x}_t'$ is a $k \times 1$ vector of covariates, β_t is the associated $k \times 1$ vector of time-varying parameters, and the disturbances ε_t^π and ε_t^h are mutually and serially uncorrelated. The log-volatility h_t follows a stationary AR(1) process with $|\phi| < 1$, and it is initialized with $h_1 \sim \mathcal{N}(\mu, \sigma^2/(1 - \phi^2))$. The model (1)–(2) allows the conditional mean of π_t to have time-varying parameters, i.e., both α_t and β_t are time-varying. The vector of coefficients $\gamma_t = (\alpha_t, \beta_t)'$ in turn evolves according to a random walk process:

$$\gamma_t = \gamma_{t-1} + \varepsilon_t^\gamma \quad \varepsilon_t^\gamma \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega}) \quad (3)$$

where $\mathbf{\Omega}$ is a $(k+1) \times (k+1)$ covariance matrix. Following the literature on time-varying parameters vector autoregressions (e.g., [Cogley and Sargent \(2005\)](#); [Cogley et al. \(2010\)](#)), we allow for a generic correlation structure among the innovations to the random-walk coefficients. Finally, the random walk process in (3) is initialized with $\gamma_1 \sim \mathcal{N}(\gamma_0, \mathbf{\Omega}_0)$ for constant matrices γ_0 and $\mathbf{\Omega}_0$.

To reduce the model in (1)–(3) into a standard TVP regression with stochastic volatility, one needs to set $\alpha_t = 0$ for all $t = 1, \dots, T$. By allowing α_t to be nonzero, the model permits an additional channel of persistence - since the log-volatility follows

an AR(1) process, a shock to h_{t-1} would affect h_t , which in turns has a direct impact on the conditional mean of π_t . This allows us to investigate how inflation uncertainty h_t affects the dynamics of inflation. Also note that (1)–(3) define a Gaussian state space model with two types of states, namely, γ_t and h_t . The model is linear in γ_t but nonlinear in h_t , and the this nonlinearity makes estimation more difficult.

Thus, to estimate the model, we follow Chan (2017) and use an efficient Markov chain Monte Carlo (MCMC) sampler. We adopt a Bayesian approach and use an MCMC algorithm to simulate from the joint posterior distribution based on band and sparse matrix algorithms.

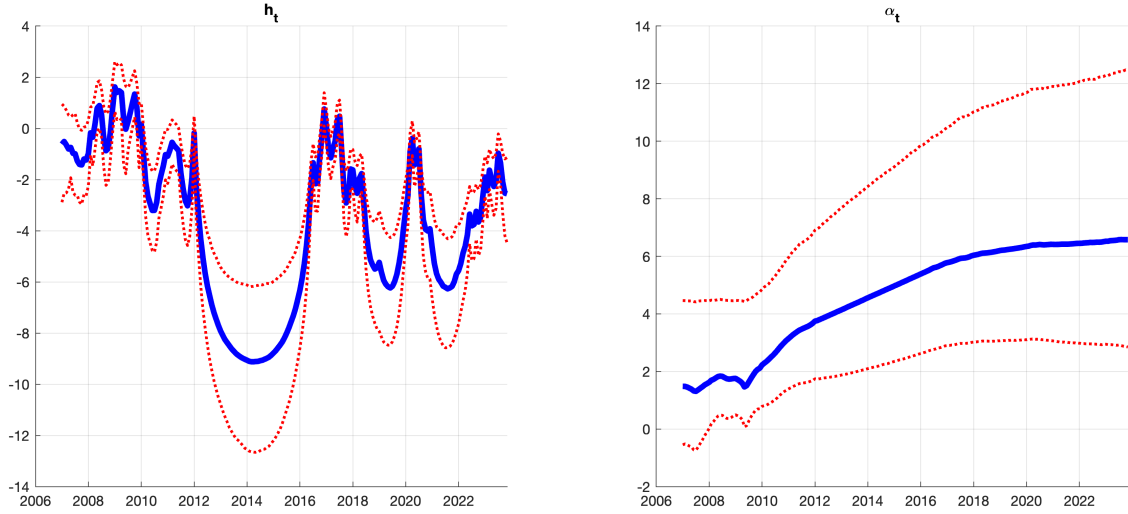
4 Empirical Results

In Figure 2 (left panel) we plot the evolution of the inflation uncertainty h_t and the associated 90% credible intervals. The estimates of h_t reveal a sharp increase in inflation volatility between January 2007 and December 2009. Subsequently, we observe a decrease in 2010, followed by a substantial decline between 2012 and 2016. Indeed, during the period from January 2007 to December 2023, the years between 2012 and 2016 are characterized by relatively low inflation volatility. These findings suggest that during this period, inflation-related uncertainty has significantly decreased.

However, we observe an increase in volatility starting from late 2016, covering the period between 2017 and 2018. Then, we observe a decrease in volatility between 2018 and the beginning of 2019. However, inflation volatility significantly increases between 2019 and 2020, before experiencing a decline during the periods marked by the Covid-19 crisis. After the crisis in 2021, inflation volatility undergoes a notable increase until the end of our sample period in December 2023.

After estimating inflation volatility in the Democratic Republic of Congo (DRC), we analyzed the impact of inflation uncertainty on inflation. In Figure 2 (right panel), we plot the evolution of α_t and the associated 90% credible intervals. There is evidently significant time variation in the estimates, underscoring the importance of the time-varying parameter extension. These findings suggest that the effects of inflation uncertainty have been persistent since 2010. For example, our results show that inflation uncertainty—measured by the stochastic volatility—has a positive impact on inflation across all of the sample starting from 2007 and the underlying impact exhibits temporal evolution.

Figure 2 – Inflation uncertainty and its impact on inflation



Source : Authors' estimation in Matlab

5 Conclusion

This study provides empirical evidence on the dynamic relationship between inflation and inflation uncertainty in the Democratic Republic of Congo (DRC) from January 2007 to December 2023, using a Time-Varying Parameter Stochastic Volatility in Mean (TVSP-SVM) framework. The findings confirm that inflation uncertainty has not only been a persistent feature of the Congolese macroeconomic landscape, but also a significant and time-dependent driver of actual inflation. Specifically, periods of elevated uncertainty—often triggered by external shocks, exchange rate instability, or policy unpredictability—have exerted a positive and amplifying effect on inflation levels. This reinforces the theoretical proposition that uncertainty does not merely accompany inflation volatility but contributes directly to it as in [Cukierman and Meltzer \(1986\)](#) and [Cukierman \(1992\)](#).

Moreover, the temporal evolution of this relationship highlights the inadequacy of static models in capturing the nuanced interplay between inflation dynamics and uncertainty in structurally fragile economies, and this has implications for policy. For the DRC, a country marked by macroeconomic volatility, weak institutional frameworks, and exposure to external shocks, this underscores the need for a more adaptive and forward-looking monetary policy framework. Policymakers must account for the

destabilizing role of uncertainty not only as a symptom of inflationary episodes but as a catalyst that can undermine long-term macroeconomic stability. Enhancing the credibility of monetary institutions, improving transparency, and developing early-warning systems that monitor volatility could significantly mitigate the inflationary risks posed by uncertainty and help anchor inflation expectations more effectively.

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Appendix : Bayesian Estimation

To complete the model specification, we assume independent priors for σ^2 , μ , ϕ and $\mathbf{\Omega}$:

$$\mu \sim \mathcal{N}(\mu_0, V_\mu), \quad \phi \sim \mathcal{N}(\phi_0, V_\phi) \mathbb{1}(|\phi| < 1), \quad \sigma^2 \sim \mathcal{IG}(\nu_{\sigma^2}, S_{\sigma^2}), \quad \mathbf{\Omega} \sim \mathcal{IW}(\nu_{\mathbf{\Omega}}, \mathbf{S}_{\mathbf{\Omega}}) \quad (4)$$

where $\mathcal{IG}(\cdot, \cdot)$ denotes the inverse-gamma distribution and $\mathcal{IW}(\cdot, \cdot)$ represents the inverse-Wishart distribution. Note that we impose the stationarity condition i.e. $|\phi| < 1$ on the prior for ϕ . For notational convenience, let $\mathbf{x}$ denote the covariates, $\pi = (\pi_1, \dots, \pi_T)'$, $\gamma = (\gamma'_1, \dots, \gamma'_T)'$ and $\mathbf{h} = (h_1, \dots, h_T)'$. Then posterior draws can be obtained by sequentially sampling from:

1. $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2, \mathbf{\Omega}) = p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$
2. $p(\gamma \mid \pi, \mathbf{x}, \mathbf{h}, \mu, \phi, \sigma^2, \mathbf{\Omega}) = p(\gamma \mid \pi, \mathbf{x}, \mathbf{h}, \mathbf{\Omega})$
3. $p(\mathbf{\Omega}, \sigma^2 \mid \pi, \mathbf{x}, \gamma, \mathbf{h}, \mu, \phi) = p(\mathbf{\Omega} \mid \gamma) p(\sigma^2 \mid \mathbf{h}, \mu, \phi)$
4. $p(\mu, \phi \mid \pi, \mathbf{x}, \gamma, \mathbf{h}, \sigma^2, \mathbf{\Omega}) = p(\mu, \phi \mid \mathbf{h}, \sigma^2)$

In Step 1 the joint conditional density $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$ is high-dimensional and nonstandard. To simulate from this density, we follow the approach in [Chan and Strachan \(2014\)](#) by exploiting the fact that the Hessian of $\log p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$ is a band matrix. As such, a Gaussian approximation can be quickly obtained, which is then used as a proposal density in an acceptance-rejection Metropolis-Hastings algorithm. Moreover, since the Hessian of this Gaussian proposal density is also a band matrix, we use the precision sampler—so called as it exploits the band structure of the precision matrix (inverse covariance matrix) of the states—to obtain candidate draws instead of Kalman filter-based algorithms.

We first discuss an efficient way to obtain a Gaussian approximation of

$$p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$$

. By Bayes' Theorem, we have

$$p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2) \propto p(\pi \mid \mathbf{x}, \gamma, \mathbf{h}) p(\mathbf{h} \mid \mu, \phi, \sigma^2)$$

In what follows, we derive explicit expressions for the two densities on the right-hand side. It turns out that the prior density $p(\mathbf{h} \mid \mu, \phi, \sigma^2)$ is Gaussian. If we approximate the likelihood $p(\pi \mid \mathbf{x}, \gamma, \mathbf{h})$ by a Gaussian density in $\mathbf{h}$, we immediately obtain a Gaussian approximation of $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$. Thus, we note that given a point $\tilde{\mathbf{h}} = (\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_T)' \in \mathbb{R}^T$, we can approximate the log conditional likelihood $\log p(\pi \mid \mathbf{x}, \gamma, \mathbf{h}) = \sum_{t=1}^T \log p(\pi_t \mid x_t, \gamma_t, h_t)$ using a second-order Taylor expansion around $\tilde{\mathbf{h}}$ to obtain

$$\begin{aligned} \log p(\pi \mid \mathbf{x}, \gamma, \mathbf{h}) &\approx \log p(\pi \mid \mathbf{x}, \gamma, \tilde{\mathbf{h}}) + (\mathbf{h} - \tilde{\mathbf{h}})' \mathbf{f} - \frac{1}{2} (\mathbf{h} - \tilde{\mathbf{h}})' \mathbf{G} (\mathbf{h} - \tilde{\mathbf{h}}) \\ &= \frac{1}{2} (\mathbf{h}' \mathbf{G} \mathbf{h} - 2 \mathbf{h}' (\mathbf{f} + \mathbf{G} \tilde{\mathbf{h}})) + c_1 \end{aligned} \quad (5)$$

where c_1 is a constant independent of $\mathbf{h}$, $\mathbf{f} = (f_1, \dots, f_T)'$ and $\mathbf{G} = \text{diag}(G_1, \dots, G_T)$ with

$$f_t = \frac{\partial}{\partial h_t} \log p(\pi_t \mid \mathbf{x}_t, \gamma_t, h_t) \big|_{h_t = \tilde{h}_t} \quad G_t = -\frac{\partial^2}{\partial h_t^2} \log p(\pi \mid \mathbf{x}_t, \gamma_t, h_t) \big|_{h_t = \tilde{h}_t}$$

In other words, $\mathbf{G}$ is the negative Hessian of the log conditional likelihood evaluated at $\tilde{\mathbf{h}}$. Also note that $\mathbf{G}$ is diagonal (hence a band matrix). Since the log-conditional density of π_t given the latent variables h_t and $\gamma_t = (\alpha_t, \beta_t)'$ is given by

$$\begin{aligned} \log p(\pi_t \mid \mathbf{x}_t, \gamma_t, h_t) &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} h_t - \frac{1}{2} e^{-h_t} (\pi_t - \mathbf{x}_t' \beta_t - \alpha_t e^{h_t})^2 \\ &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} h_t - \frac{1}{2} (\alpha_t^2 e^{h_t} + e^{-h_t} (\pi_t - \mathbf{x}_t' \beta_t)^2 - 2\alpha_t (\pi_t - \mathbf{x}_t' \beta_t)) \end{aligned}$$

One can write that

$$\begin{aligned} \frac{\partial}{\partial h_t} \log p(\pi_t \mid \mathbf{x}_t, \gamma_t, h_t) &= -\frac{1}{2} - \frac{1}{2} \alpha_t^2 e^{h_t} - \frac{1}{2} e^{-h_t} (\pi_t - \mathbf{x}_t' \beta_t)^2, \\ \frac{\partial^2}{\partial h_t^2} \log p(\pi_t \mid \mathbf{x}_t, \gamma_t, h_t) &= -\frac{1}{2} \alpha_t^2 e^{h_t} - \frac{1}{2} e^{-h_t} (\pi_t - \mathbf{x}_t' \beta_t)^2 \end{aligned}$$

Next, we derive the prior density $p(\mathbf{h} \mid \mu, \phi, \sigma^2)$. To that end, let

$$\mathbf{H}_\phi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\phi & 1 & 0 & \cdots & 0 \\ 0 & -\phi & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\phi & 1 \end{pmatrix}.$$

We note that the determinant of $\mathbf{H}_\phi$ is 1—hence, it is invertible—regardless of the value of ϕ . Also observe that $\mathbf{H}_\phi$ is a lower bidiagonal matrix, i.e., its nonzero elements are confined along the main diagonal and the diagonal below. Now, write the state equation of h_t in (2) as:

$$\mathbf{H}_\phi \mathbf{h} = \tilde{\delta}_h + \varepsilon^h \quad \varepsilon^h \sim \mathcal{N}(0, \mathbf{S}_h)$$

where $\tilde{\delta}_h = (\mu, (1 - \phi)\mu, \dots, (1 - \phi)\mu)'$, $\varepsilon^h = (\varepsilon_1^h, \dots, \varepsilon_T^h)'$ and $\mathbf{S}_h = \text{diag}(\sigma^2/(1 - \phi^2), \sigma^2, \dots, \sigma^2)$. That is, $(\mathbf{h} \mid \mu, \phi, \sigma^2) \sim \mathcal{N}(\delta_h, (\mathbf{H}'_\phi \mathbf{S}_h^{-1} \mathbf{H}_\phi)^{-1})$ with log-density

$$\log p(\mathbf{h} \mid \mu, \phi, \sigma^2) = -\frac{1}{2}(\mathbf{h}' \mathbf{H}'_\phi \mathbf{S}_h^{-1} \mathbf{H}_\phi \mathbf{h} - 2\mathbf{h}' \mathbf{H}'_\phi \mathbf{S}_h^{-1} \mathbf{H}_\phi \tilde{\delta}_h) + c_2 \quad (6)$$

where $\delta_h = \mathbf{H}'_\phi^{-1} \tilde{\delta}_h$ and c_2 is a constant independent of $\mathbf{h}$. Finally, combining (5) and (6), we have

$$\begin{aligned} p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2) &= \log p(\pi \mid \mathbf{x}, \gamma, \mathbf{h}) + \log p(\mathbf{h} \mid \mu, \phi, \sigma^2) + c_3 \\ &\approx -\frac{1}{2}(\mathbf{h}' \mathbf{K}_h \mathbf{h} - 2\mathbf{h}' \mathbf{k}_h) + c_4 \end{aligned} \quad (7)$$

where c_3 and c_4 are constants independent of $\mathbf{h}$, $\mathbf{K}_h = \mathbf{H}'_\phi \mathbf{S}_h^{-1} \mathbf{H}_\phi + \mathbf{G}$ and $\mathbf{k}_h = \mathbf{f} + \mathbf{G} \tilde{\mathbf{h}} + \mathbf{H}'_\phi \mathbf{S}_h^{-1} \mathbf{H}_\phi \tilde{\delta}_h$. It can be shown that the expression in (7) is the log-kernel of the $\mathcal{N}(\hat{\mathbf{h}}, \mathbf{K}_h^{-1})$ density, where $\hat{\mathbf{h}} = \mathbf{K}_h^{-1} \mathbf{k}_h$. In other words, $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$ can be approximated by the Gaussian density with mean vector $\hat{\mathbf{h}}$ and precision matrix $\mathbf{K}_h$. For our purpose it is important to realize that the precision matrix $\mathbf{K}_h$ is tridiagonal—i.e., its nonzero elements appear only on the main diagonal and the diagonals above and below the main one. One consequence is that $\hat{\mathbf{h}}$ can be computed efficiently by solving the linear system $\mathbf{K}_h \mathbf{x} = \mathbf{k}_h$ for $\mathbf{x}$ without calculating the

inverse $\mathbf{K}_{\mathbf{h}}^{-1}$. In addition, draws from $\mathcal{N}(\hat{\mathbf{h}}, \mathbf{K}_{\mathbf{h}}^{-1})$ can be quickly obtained using the precision sampler in [Chan and Jeliazkov \(2009\)](#). This Gaussian approximation is then used as the proposal density in the acceptance-rejection Metropolis-Hastings step. It remains to select the point $\tilde{\mathbf{h}}$ for the Taylor expansion in (5). We choose $\tilde{\mathbf{h}}$ to be the mode of $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$, which can be quickly obtained by the Newton-Raphson method (see below).

In order to implement Step 2, first note that we can rewrite (1) as

$$\pi_t = \mathbf{z}_t' \gamma_t + \varepsilon_t^\pi \quad \varepsilon_t^\pi \sim \mathcal{N}(0, e^{h_t}) \quad (8)$$

where $\mathbf{z} = (\exp(h_t), \mathbf{z}_t')'$ and $\gamma_t = (\alpha_t, \beta_t)'$. Then (3) and (8) define a linear Gaussian state space model in γ_t . Hence, to draw from $p(\gamma \mid \pi, \mathbf{x}, \mathbf{h}, \boldsymbol{\Omega})$ and we follow an efficient precision sampler ([Chan and Jeliazkov \(2009\)](#); [McCausland et al. \(2011\)](#)). Lastly, Step 3 to Step 4 are standard.

Let first discuss how one can quickly locate the maximum of $p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$ or equivalently that of $\log p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$, using the Newton-Raphson method. It follows from (14) and (15) that the negative Hessian of $\log p(\mathbf{h} \mid \pi, \mathbf{x}, \gamma, \mu, \phi, \sigma^2)$ evaluated at $\mathbf{h} = \tilde{\mathbf{h}}$ is $\mathbf{K}_{\mathbf{h}}$ and the gradient at $\mathbf{h} = \tilde{\mathbf{h}}$ is $-\mathbf{K}_{\mathbf{h}}\tilde{\mathbf{h}} + \mathbf{k}_{\mathbf{h}}$. Hence, we can implement the Newton-Raphson method as follows: initialize with $\mathbf{h} = \mathbf{h}^{(1)}$ for some constant vector $\mathbf{h}^{(1)}$. For $t = 1, 2, \dots$, use $\tilde{\mathbf{h}} = \mathbf{h}^{(t)}$ in the evaluation of $\mathbf{K}_{\mathbf{h}}$ and $\mathbf{k}_{\mathbf{h}}$, and compute

$$\mathbf{h}^{(t+1)} = \mathbf{h}^{(t)} + \mathbf{K}_{\mathbf{h}}^{-1}(-\mathbf{K}_{\mathbf{h}}\mathbf{h}^{(t)} + \mathbf{k}_{\mathbf{h}}) = \mathbf{K}_{\mathbf{h}}^{-1}\mathbf{k}_{\mathbf{h}}$$

Repeat this procedure until some convergence criterion is reached, e.g., when $\|\mathbf{h}^{(t+1)} - \mathbf{h}^{(t)}\| < \varepsilon$ for some prefixed tolerance level ε .

Next, we discuss Step 2 of the sampler: to efficiently sample from $p(\gamma \mid \pi, \mathbf{x}, \mathbf{h}, \boldsymbol{\Omega})$. Stacking (8) over $t = 1, \dots, T$, we have

$$\pi_t = \mathbf{Z}\gamma + \varepsilon^\pi \quad \varepsilon_t \sim \mathcal{N}(0, \mathbf{S}_\pi) \quad (9)$$

where $\varepsilon^\pi = (\varepsilon_1^\pi, \dots, \varepsilon_T^\pi)'$,

$$\mathbf{S}_y = \begin{pmatrix} e^{h_1} & 0 & \cdots & 0 \\ 0 & e^{h_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{h_T} \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} \mathbf{z}'_1 & 0 & \cdots & 0 \\ 0 & \mathbf{z}'_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{z}'_T \end{pmatrix}.$$

To derive the prior density of γ implied by (3), rewrite the state equation of γ_t in matrix notations:

$$\mathbf{H}\gamma = \tilde{\delta}_\gamma + \varepsilon^\gamma \quad \varepsilon_t \sim \mathcal{N}(0, \mathbf{S}_\gamma)$$

where $\tilde{\delta}_\gamma = (\gamma'_0, 0, \dots, 0)'$, $\varepsilon^\gamma = (\varepsilon_1^\gamma, \dots, \varepsilon_T^\gamma)'$, $\mathbf{S}_\gamma = \text{diag}(\mathbf{\Omega}_0, \mathbf{\Omega}, \dots, \mathbf{\Omega})$ and $\mathbf{H}$ is the first difference matrix.

$$\mathbf{H} = \begin{pmatrix} I_{k+1} & 0 & 0 & \cdots & 0 \\ -I_{k+1} & I_{k+1} & 0 & \cdots & 0 \\ 0 & -I_{k+1} & I_{k+1} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -I_{k+1} & I_{k+1} \end{pmatrix}.$$

That is, $(\gamma \mid \mathbf{\Omega}) \sim \mathcal{N}(\delta_\gamma, (\mathbf{H}\mathbf{S}_\gamma^{-1}\mathbf{H})^{-1})$, where $\delta_\gamma = \mathbf{H}^{-1}\tilde{\delta}_\gamma$. Now, using standard results from linear regression (see, e.g., [Kroese et al. \(2014\)](#), Corollary 8.1), we have,

$$(\gamma \mid \pi, \mathbf{h}, \mathbf{\Omega}) \sim \mathcal{N}(\hat{\gamma}, \mathbf{D}_\gamma)$$

where $\mathbf{D}_\gamma^{-1} = \mathbf{H}'\mathbf{S}_\gamma^{-1}\mathbf{H} + \mathbf{Z}'\mathbf{S}_\pi^{-1}\mathbf{Z}$ and $\hat{\gamma} = (\mathbf{H}'\mathbf{S}_\gamma^{-1}\mathbf{H}\delta_\gamma + \mathbf{Z}'\mathbf{S}_\pi^{-1}\pi)$.

Since $\mathbf{D}_\gamma$ is typically a high-dimensional full covariance matrix, the conventional sampling approach that requires the Cholesky factor of $\mathbf{D}_\gamma$ is time-consuming. However, since the precision $\mathbf{D}_\gamma^{-1}$ is a band matrix, one can efficiently sample from $\mathcal{N}(\hat{\gamma}, \mathbf{D}_\gamma)$ using band and sparse matrix routines; we refer the readers to [Chan and Jeliazkov \(2009\)](#) for details.

To implement Step 3, first note that $\mathbf{\Omega}$ and σ^2 are conditionally independent given the latent states γ and $\mathbf{h}$. In fact, both conditional distributions are standard distributions:

$$(\mathbf{\Omega} \mid \gamma) \sim \mathcal{IW}(\nu_\mathbf{\Omega} + T - 1, \mathbf{S}_\mathbf{\Omega} + \Sigma_{t=2}^T(\gamma_t - \gamma_{t-1})(\gamma_t - \gamma_{t-1})')$$

,

$$(\sigma^2 \mid \mathbf{h}, \mu, \phi) \sim \mathcal{IG}\left(\nu_{\sigma^2} + \frac{T}{2}, \tilde{S}_{\sigma^2}\right)$$

where $\tilde{S}_{\sigma^2} = S_{\sigma^2} + ((1 - \phi^2)(h_1 - \mu)^2 + \sum_{t=2}^T (h_t - \mu - \phi(h_{t-1} - \mu))^2)/2$.

Lastly, we sample μ and ϕ jointly to improve efficiency. Specifically, we implement an independence-chain Metropolis-Hastings step with a bivariate Student-t proposal with degree of freedom parameter set to 5. Recall that ϕ is bounded between -1 and 1 . For computational convenience, we consider the transformation $\eta = \tanh(\phi)$ so that η takes any real value. Then the mode of the joint density of (μ, η) can be obtained using the Newton-Raphson method.

Now, we provide the details of the hyper-parameters used to estimate h_t and α_t . The prior distributions are described in (4). In particular, μ follows a Gaussian distribution with mean $\mu_0 = 0$ and variance $V_\mu = 10$; ϕ has a truncated Gaussian distribution with $\phi_0 = 0.97$ and $V_\phi = 0.1^2$. In addition, β has a normal prior $\mathcal{N}(\beta_0, V_\beta)$ with $\beta_0 = 0$ and $V_\beta = 10$.

The variance parameters are independently distributed as $\sigma^2 \sim \mathcal{IG}(\nu_{\sigma^2}, V_{\sigma^2})$ and $\mathbf{\Omega} \sim \mathcal{IW}(\nu_{\mathbf{\Omega}}, S_{\mathbf{\Omega}})$. We choose relatively small—hence relatively noninformative—values for the degrees of freedom parameters: $\nu_{\sigma^2} = \nu_{\mathbf{\Omega}} = 10$. For the scale parameters, we set $S = 0.36$ and $\mathbf{S}_{\mathbf{\Omega}} = \text{diag}(0.13, 0.8125)$. These values imply $\mathbb{E}\sigma^2 = 0.2^2$, $\mathbb{E}\mathbf{\Omega} = \text{diag}(0.12, 0.252)$. The chosen prior means reflect the desired smoothness of the corresponding state transition, and are comparable to those used in previous studies, such as Chan (2013) and Stock and Watson (2007).