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Weak Solutions of Conservation Laws : An Application to Inviscid Burger's Equations

Bien-Aimé Kiakisolako*

Abstract

Modeling natural phenomena is an important problem; however, it can be difficult due to factors that can create discontinuities making it difficult to find the expected solutions to the models. Hence, we need to look for other types of solution with less restrictive conditions on their differentiability. As uniqueness under these less restrictive conditions is not guaranteed, conditions from gas dynamics are used to pick the more physically realistic solution as the exact solution.

Keywords: Conservation Laws, Inviscid Burger's Equations.

1 Introduction

The study of gas motion is governed by a set of Partial Differential Equations (PDEs) called Euler equations (1). These equations correspond to the Navier-Stokes equations (2) without viscosity or deviatoric stress tensor ($\mu = 0$) and thermal conductivity. These are derived from conservation laws (conservation of momentum, of mass, of energy, etc).

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \cdot \mathbf{u})\mathbf{u} = -\nabla \cdot \mathbf{w} + \mathbf{f} \quad (1)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\nabla \cdot \mathbf{u})\mathbf{u} \right) = -\nabla p + \nabla \cdot \left\{ \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbf{I} \right] \right\} + \rho \mathbf{f}. \quad (2)$$

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where $\mathbf{u}$ is the velocity of the flow, $\nabla \cdot$ is the divergence, $\mathbf{I}$ is the identity tensor, ρ is the density, p is the pressure, $\mathbf{f}$ is the body acceleration, μ is the linear elasticity, and ω is the thermodynamic work.

However, due to the formation of discontinuities (shock waves, contact discontinuities, etc.), classical solutions to Euler equations can fail to exist. For this reason, we need to extend the notion of solution allowing solutions that satisfy less strong conditions. These will be called weak solutions.

This paper focuses on the mathematics behind weak solutions and an application in gas dynamics. After an introduction to weak solutions of single conservation laws, considering from time to time the special case of the inviscid Burger's equation $u_t + uu_x = 0$, weak solutions are presented as an extension of classical solutions. As uniqueness of weak solutions is not guaranteed, we introduce the Rankine-Hugoniot jump condition and additional conditions to pick the more physically realistic solution. As shocks can form from smooth solutions, the notion of breaking time also gives us important information to predict shock formation.

2 Conservation Laws

Let's consider a one-dimensional medium M containing a gas (or another substance that can move) and let Q denote its quantity. Suppose $u(x, t)$ measures the density of this gas at position x of M at time t .

If T is a small segment of M from $x = a$ to $x = b$ ($a \in \mathbb{R}$, $b \in \mathbb{R}$ with $a < b$), it is assumed that changes in the quantity of the gas in T can only occur in the following three ways:

- The gas enters in T at position $x = a$;
- The gas is created or destroyed inside of T ;
- The gas leaves T through position $x = b$.

Quantifying the different parts of this process, yields the following equation known as the conservation law (in integral form):

$$\frac{d}{dt} \int_a^b u(x, t) dx = \phi(a, t) - \phi(b, t) + \int_a^b f(x, t) dx. \quad (3)$$

In the above equation, $\phi(x, t)$ is the flux, which is the rate at which the gas flows through position x at time t (at (x, t)) and $f(x, t)$ is the source term interpreting the rate of created or destroyed gas in the medium at (x, t) .

When u and ϕ have continuous first derivatives, equation (3) is written as

$$\int_a^b (u_t(x, t) + \phi_x(x, t) - f(x, t)) dx = 0.$$

This integral is zero for all reals $a < b$ along the medium M , and so we deduce the following differential form:

$$u_t(x, t) + \phi(x, t)_x = f(x, t). \quad (4)$$

Often, an additional equation relating u and ϕ is given, for example ϕ can depend on u , so that equation (4) becomes

$$u_t + [\phi(u)]_x = f \quad (5)$$

which has the form of equation (1).

3 Weak Solutions

Given the following initial-value problem for the k th order PDE

$$\begin{cases} F(\vec{x}, u, Du, \dots, D^k u) = 0, & \vec{x} \in \mathbb{R}^n, \ t > 0 \\ u(\vec{x}, 0) = u_0(x) \end{cases} \quad (6)$$

a **classical** solution $u(\vec{x}, t)$ is a k times continuously differentiable function satisfying (6).

For the following initial value problem of (5), with the source term $f(\vec{x}, t) = 0$:

$$\begin{cases} u_t + [\phi(u)]_x = 0, & t > 0, \ -\infty < x < \infty \\ u(x, 0) = u_0(x) \end{cases} \quad (7)$$

where $u_0(x)$ is continuous, a classical solution $u(x, t)$ is a function such that for all $x \in \mathbb{R}$ and $t > 0$: u is continuous, u_x and u_t exist and are continuous, u satisfies (7) and $u(x, 0) = u_0(x)$.

Weak solutions allow us to consider functions that may not even be continuous, hence not differentiable, as solutions in the way explained below.

Multiplying a function $v : \mathbb{R} \times [0, \infty) \mapsto \mathbb{R}$, smooth with compact support (**test function**), by a classical solution u of (7) and integrating over $\mathbb{R} \times [0, \infty)$. This yields:

$$\begin{aligned}
& \int_0^\infty \int_{-\infty}^{+\infty} [u_t v + [\phi(u)]_x v] dx dt = 0 \\
& \Rightarrow \int_0^\infty \int_{-\infty}^\infty u_t v dx dt + \int_0^\infty \int_{-\infty}^\infty [\phi(u)]_x v dx dt = 0 \\
& \Rightarrow T_1 + T_2 = 0.
\end{aligned} \tag{8}$$

Integrating by parts, the first integral T_1 yields:

$$\begin{aligned}
T_1 &= \int_{-\infty}^\infty \left[\int_0^\infty u_t v dt \right] dx \\
&= \int_{-\infty}^\infty \left[uv \Big|_0^\infty - \int_0^\infty uv_t dt \right] dx \\
&= \int_{-\infty}^\infty \left[-u(x, 0)v(x, 0) - \int_0^\infty uv_t dt \right] dx \quad (\text{as } v(x, t) \text{ vanishes at infinity}) \\
&= \int_{-\infty}^\infty \left[-u_0(x)v(x, 0) - \int_0^\infty uv_t dt \right] dx \\
&= - \int_{-\infty}^\infty u_0(x)v(x, 0) dx - \int_{-\infty}^\infty \int_0^\infty uv_t dx dt.
\end{aligned}$$

Thus

$$T_1 = - \int_{-\infty}^\infty u_0(x)v(x, 0) dx - \int_{-\infty}^\infty \int_0^\infty uv_t dx dt. \tag{9}$$

Integrating by parts, the second integral T_2 yields:

$$\begin{aligned}
T_2 &= \int_0^\infty \left[\int_{-\infty}^\infty [\phi(u)]_x v \, dx \right] dt \\
&= \int_0^\infty \left[[\phi(u)]v \Big|_{-\infty}^\infty - \int_{-\infty}^\infty [\phi(u)]v_x \, dx \right] dt \\
&= - \int_0^\infty \int_{-\infty}^\infty [\phi(u)]v_x \, dx dt \quad (\text{as } v(x, t) \text{ vanishes at infinity}).
\end{aligned}$$

Thus

$$T_2 = - \int_{-\infty}^\infty \int_0^\infty [\phi(u)](x, t) v_x(x, t) \, dx dt \quad (10)$$

Putting (9) and (10) together, (8) becomes:

$$\int_{-\infty}^\infty u_0(x) v(x, 0) dx + \int_{-\infty}^\infty \int_0^\infty [uv_t + [\phi(u)]v_x] \, dx dt = 0. \quad (11)$$

Equation (11) is the weak form of (7).

It is clear that in the weak form (11), u_t or u_x do not appear explicitly.

A weak solution of (7) is a function $u(x, t)$ that satisfies (11) for every test function $v : \mathbb{R} \times [0, \infty)$.

From the above discussion, it follows that a classical solution of (7) is also a weak solution of (7). This means that weak solutions are an extension of classical solutions.

4 Shock Waves and Rankine-Hugoniot Condition

Consider an initial value problem (7) with a weak solution $u(x, t)$ piecewise smooth on the left and right sides of a curve $x = \xi(t)$ of discontinuity. If the method of characteristics gives classical solutions on each side, we say that $u(x, t)$ has a **jump** discontinuity across the curve $x = \xi(t)$.

While different discontinuity curves can exist, the conservation law will intrinsically select only one verifying the following jump condition.

4.1 The Rankine-Hugoniot Jump Condition

Theorem 1 (Rankine-Hugoniot Jump Condition). If a weak solution u of (7), discontinuous across the curve $x = \xi(t)$, is smooth on both sides of $x = \xi(t)$ then u

satisfies

$$\frac{\phi(u^-) - \phi(u^+)}{u^- - u^+} = \xi'(t) \quad (12)$$

across $x = \xi(t)$, with $u^+(x, t)$ and $u^-(x, t)$ the limits of u approaching $x = \xi(t)$ from the right and from the left, respectively.

For every test function $v : \mathbb{R} \times [0, \infty)$, we know that the weak solution $u(x, t)$ of (7) satisfies

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi(u)v_x] \, dxdt + \int_{-\infty}^\infty \phi(x)v(x, 0) \, dx = 0.$$

In particular, for $v(x, t)$ verifying $v(x, 0) = 0$, this equations simplifies to

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi(u)v_x] \, dxdt = 0. \quad (13)$$

Let Γ^- be the region on the left of $\xi(t)$ and Γ^+ the one on the right, ie

$$\Gamma^- = \{(x, t) : t > 0, -\infty < x < \xi(t)\}$$

$$\Gamma^+ = \{(x, t) : t > 0, \xi(t) < x < +\infty\}.$$

We have:

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi(u)v_x] \, dxdt = \int \int_{\Gamma^-} [uv_t + \phi(u)v_x] \, dxdt + \int \int_{\Gamma^+} [uv_t + \phi(u)v_x] \, dxdt$$

(13) implies that

$$\int \int_{\Gamma^-} [uv_t + \phi(u)v_x] \, dxdt + \int \int_{\Gamma^+} [uv_t + \phi(u)v_x] \, dxdt = 0. \quad (14)$$

On the left side of the discontinuity curve, we have:

$$\begin{aligned}
& \int \int_{\Gamma^-} [uv_t + \phi(u)v_x] dx dt \\
&= \int_0^\infty \int_{x=\xi(t)} [u_t^- + (\phi(u^-))_x] v \, dx dt + \int \int_{\Gamma^-} [uv_t + \phi(u)v_x] \, dx dt \\
&= \int_0^\infty \int_{x=\xi(t)} [u_t^- + [\phi(u^-)]_x] v \, dx dt + \int_{-\infty}^{x=\xi(t)} \left[\int_0^\infty uv_t \, dt \right] dx + \int_0^\infty \left[\int_{-\infty}^{x=\xi(t)} \phi(u)v_x \, dx \right] dt \\
&= \int_0^\infty \int_{x=\xi(t)} [u_t^- + (\phi(u^-))_x] v \, dx dt - \int_{-\infty}^{x=\xi(t)} \int_0^\infty u_t v \, dx dt - \int_0^\infty \int_{-\infty}^{x=\xi(t)} [\phi(u)]_x v \, dx dt \\
&= \int_0^\infty \int_{x=\xi(t)} (u_t^- v + [\phi(u^-)]_x v) \, dx dt - \int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v \, dx dt \\
&= \int_0^\infty \int_{x=\xi(t)} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \cdot (\phi(u^-), u^-) v \, dx dt - \int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v \, dx dt \\
&= \int_{x=\xi(t)} v(\phi(u^-), u^-) \cdot (\nu_1, \nu_2) \, ds - \int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v \, dx dt \\
&= \int_{x=\xi(t)} (u^- v \nu_2 + \phi(u^-) v \nu_1) \, ds - \int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v \, dx dt
\end{aligned}$$

with $\nu = (\nu_1, \nu_2)$ the outward unit normal to Γ^- . Hence,

$$\int \int_{\Gamma^-} [uv_t + \phi(u)v_x] \, dx dt = \int_{x=\xi(t)} (u^- v \nu_2 + \phi(u^-) v \nu_1) \, ds - \int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v \, dx dt. \quad (15)$$

On the right side of the discontinuity curve, we have:

$$\int \int_{\Gamma^+} [uv_t + \phi(u)v_x] \, dx dt = - \int_{x=\xi(t)} (u^+ v \nu_2 + \phi(u^+) v \nu_1) \, ds - \int \int_{\Gamma^+} [u_t + [\phi(u)]_x] v \, dx dt \quad (16)$$

with $\nu' = (-\nu_1, -\nu_2)$ the outward unit normal to Γ^+ .

In (15) and (16),

$$\int \int_{\Gamma^-} [u_t + [\phi(u)]_x] v dx dt = 0 \text{ and } \int \int_{\Gamma^+} [u_t + [\phi(u)]_x] v dx dt = 0$$

as $u(x, t)$ is a classical solution on Γ^- and on Γ^+ respectively.

Thus, for every test function $v(x, t)$ (14) becomes

$$\int_{x=\xi(t)} [u^- \nu_2 + \phi(u^-) \nu_1] v ds = \int_{x=\xi(t)} [u^+ \nu_2 + \phi(u^+) \nu_1] v ds.$$

Hence $u^- \nu_2 + \phi(u^-) \nu_1 = u^+ \nu_2 + \phi(u^+) \nu_1$, then

$$\frac{\phi(u^-) - \phi(u^+)}{u^- - u^+} = -\frac{\nu_2}{\nu_1}. \quad (17)$$

The slope of ν , the normal to $x = \xi(t)$, being $\frac{\nu_1}{\nu_2}$, the slope of $x = \xi(t)$ is given by

$$\xi'(t) = -\frac{\nu_2}{\nu_1}.$$

Thus, (17) becomes

$$\xi'(t) = \frac{\phi(u^-) - \phi(u^+)}{u^- - u^+}. \quad (18)$$

Consequently, if a weak solution has a discontinuity across a curve, then it must verify (18) called the Rankine-Hugoniot Jump Condition.

$\xi'(t)$ is the speed of the discontinuity curve, or the shock speed. $[u]$ and $[\phi(u)]$ are the jumps of u and $\phi(u)$ across the discontinuity curve $x = \xi(t)$, respectively.

4.2 Shock Wave Solutions

A piecewise smooth solution $u(x, t)$ of (7) with a jump across a discontinuity curve $x = \xi(t)$ satisfying the Rankine-Hugoniot Jump Condition (18) is called a **shock wave solution**.

We now consider the Burger's equation $u_t + uu_x = 0$ with initial data

$$u_0(x) = \begin{cases} u^- & \text{for } x < 0 \\ u^+ & \text{for } x > 0 \end{cases}$$

with $u^- > u^+$, the associated characteristic equations give as speed of the solution $\frac{dx}{dt} = u$ meaning that the taller the wave, the faster it moves. Then the wave on the right is expected to be overtaken by the one on the left causing the wave to break and resulting in the discontinuity curve.

An initial value problem (7) can have more than one solution, we choose the more physically relevant as the real solution. This is done by only allowing discontinuity curves $x = \xi(t)$ such that the wave on the left moves faster than the wave on the right, ie

$$\phi'(u^-) > \xi'(t) > \phi'(u^+) \quad (19)$$

(19) is known as the **entropy condition**.

4.3 Definition (Shock Curves)

Given a weak solution u to (7), a shock curve for u is a discontinuity curve that satisfies the entropy condition and Rankine-Hugoniot jump condition.

In (7), if ϕ is uniformly convex, a Riemann problem has a unique weak admissible solution:

If $u^- > u^+$, the admissible solution along the shock curve $x = \sigma t$ is given by

$$u(x, t) = \begin{cases} u^- & x < \sigma t \\ u^+ & x > \sigma t \end{cases}$$

with σ given by the Rankine-Hugoniot jump condition.

If $u^- < u^+$, there exists a continuous solution given by

$$u(x, t) = \begin{cases} u^- & x < \phi'(u^-)t \\ (\phi')^{-1}\left(\frac{x}{t}\right) & \phi'(u^-)t < x < \phi'(u^+)t \\ u^+ & x > \phi'(u^+)t \end{cases}$$

Proof. If $u^- > u^+$, it is clear given the discontinuity curve $x = \sigma t$ that the Rankine-Hugoniot jump condition is verified. Moreover, as ϕ is strictly convex, $u^- > u^+$ ensures that u satisfies the entropy condition.

If $u^- < u^+$, the only thing to check is whether u is a classical solution of the Riemann' problem for $\phi'(u^-)t < x < \phi'(u^+)t$.

$$\begin{aligned}
u_t + [\phi(u)]_x &= \left[(\phi')^{-1} \left(\frac{x}{t} \right) \right]_t + \left[\phi \left((\phi')^{-1} \left(\frac{x}{t} \right) \right) \right]_x \\
&= [(\phi')^{-1}]' \left(-\frac{x}{t^2} \right) + \phi' \left((\phi')^{-1} \left(\frac{x}{t} \right) \right) [(\phi')^{-1}]' \left(\frac{x}{t} \right) \left(\frac{1}{t} \right) \\
&= - [(\phi')^{-1}]' \left(\frac{x}{t^2} \right) + [(\phi')^{-1}]' \left(\frac{x}{t^2} \right) \\
&= 0.
\end{aligned}
\tag*{$\square$}$$

4.4 Oleinik Entropy Condition

There are situations where the flux ϕ may not be uniformly convex. In this case, we want to allow discontinuity with slope σ such that $\phi'(u^-) \geq \sigma \geq \phi'(u^+)$ but without uniform convexity, uniqueness of solutions is not guaranteed.

Thus we need an alternative condition to guarantee uniqueness of solutions of (7).

When ϕ is uniformly convex, the entropy condition (19) is equivalent to

$$\frac{\phi(u^-) - \phi(u)}{u^- - u} \geq \frac{\phi(u^-) - \phi(u^+)}{u^- - u^+} \tag{20}$$

for all u with $u^+ < u < u^-$. However, this equivalence is not true when ϕ is not uniformly convex but (20) always implies (19) making it a stronger condition known as the Oleinik entropy condition.

— A shock that satisfies the Oleinik entropy condition is called an admissible shock. — An admissible weak solution is called admissible entropy if all shocks are admissible in the sense given above.

Notice that u^- is not necessarily greater than u^+ for the Oleinik entropy condition to hold.

4.5 Example

Consider the inviscid Burger's equation $u_t + uu_x = 0$, $x \in \mathbb{R}$, $t > 0$ along with the initial condition

$$u(x, 0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1. \end{cases} \tag{21}$$

We aim to find a continuous solution for $t < 1$.

For $r \in \mathbb{R}$, by the method of characteristics, $u(x, t)$ is constant on the characteristic given by $x(r, s) = ut + r$.

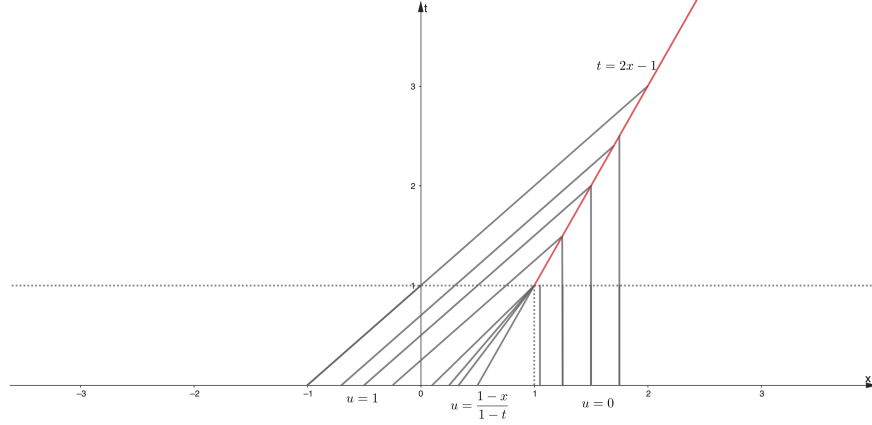


Figure 1 – Characteristic diagram of the Burger's equation in example 21

- For $r < 0$, $u(r, 0) = 1$, the characteristic are $t = x - r$.
- For $0 < r < 1$, $u(r, 0) = 1 - r$ along the characteristics $t = \frac{x}{1-r} - \frac{r}{1-r}$.
- For $r > 1$, $u(r, 0) = 0$ along the with characteristics $x = r$.

Thus, for $t < 1$, the continuous solution is

$$u(x, t) = \begin{cases} 1 & x < t \\ \frac{1-x}{1-t} & t < x < 1 \\ 0 & x > 1. \end{cases}$$

Characteristics intersect at $(1, 1)$, let's fit the shock and find the form of the solution for $t > 1$. By the Rankine-Hugoniot jump condition, the slope of the discontinuity curve is $\sigma = \frac{\phi(u^-) - \phi(u^+)}{u^- - u^+}$ where ϕ , the flux is $\phi(u) = \frac{u^2}{2}$, $u^- = 1$ and $u^+ = 0$ so that $\sigma = \frac{1}{2}$. The equation of the discontinuity curve is $x = \frac{t+1}{2}$ so that, for $t > 1$, the solution (weak) is given by

$$u(x, t) = \begin{cases} 1 & x < \frac{t+1}{2} \\ 0 & x > \frac{t+1}{2}. \end{cases}$$

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