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Towards a Formal and Constructive Representation for Irrational Numbers

Jean-Paul K. Tsasa*

Abstract

The paper addresses a simple issue: how can the set of irrational numbers, $\text{ir}\mathbb{Q}$, be constructively represented? Here is our response:

$$\text{ir}\mathbb{Q} := \left\{ \frac{p}{q} + \epsilon : (p, q) \in \mathbb{Z} \times \mathbb{Z}^*, \epsilon \sim \text{p.i} \right\},$$

where ϵ is an irrational perturbation. We show how to carefully characterize the term ϵ and discuss the didactic value of our approach.

Keywords: Set, Irrational numbers.

1 Introduction

The set of irrational numbers, $\text{ir}\mathbb{Q}$, although well-known and widely studied, remains a fascinating mathematical object, particularly when it comes to formal representation¹. Irrational numbers play a key role in many fields, from mathematical analysis to social sciences. However, they cannot be expressed as a fraction of two integers or as a periodic decimal expansion. Thus, their formal characterization remains an interesting conceptual challenge.

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1. Here are two remarks: First, the proposed notation, $\text{ir}\mathbb{Q}$, is inspired by the convention used for the set of purely imaginary numbers. Second, a set can be specified in two ways (see e.g. [Grami 2022](#), pp. 67-68): by roster method or by using set-builder notation. The roster method lists all the elements in the set. For example, the set of prime numbers could be described using braces like this: $A = \{2, 3, 5, 7, 11, \dots\}$. The ellipsis “...” simply indicates that the A is an infinite set. Alternatively, the set-builder notation describes a set using the following format: $B = \{x : x \text{ is a prime number}\}$. It reads as B is the set of all x such that x is a prime number.

Several recent mathematical contributions have explored various methods for representing irrational numbers, including Cauchy sequences, Dedekind cuts, standard/dual Baire sequences, standard Sturmian word, decimal expansions, binary expansions, and infinite continued fraction².

To the best of our knowledge, we have not found any textbook or document that formally and constructively represents the set of irrational numbers, carefully using either the roster method or set-builder notation. But we have found examples for the set of rational numbers (see, for instance, [Bick 1971](#), [Bouvier et al. 2013](#), [Bonheur 2019](#)). This note introduces a formal approach to constructively characterize the set of irrational numbers using a formalization based on a rational-*irrational perturbation* decomposition³.

Our purpose is to provide an elegant and intuitive perspective that not only meets formal requirements but also illustrates the peculiar nature of these numbers. Therefore, our study takes a pedagogical approach, aiming to introduce irrational numbers progressively, starting from known subsets, notably $\mathbb{N}$ and $\mathbb{Z}$, without resorting to larger sets such as $\mathbb{R}$ and $\mathbb{C}$.

The rest of the paper is organized as follows. We begin by recalling the fundamental elements needed for our characterization strategy ([Section 2](#)), followed by some numerical examples to illustrate the proposed concepts ([Section 3](#)), and a concluding remark ([Section 4](#)). This work aims to enrich our pedagogical experience in teaching mathematics while providing a solid foundation for further investigations.

2 Characterization

It is common to encounter mathematics textbooks presenting standard numerical sets either by roster method, as is the case, for example, with the set of integers, $\mathbb{Z} \equiv \{\dots, -2, -1, 0, 1, 2, \dots\}$, or using set-builder notation. For instance, using set-builder notation, the set of rational numbers can be expressed as:

$$\mathbb{Q} := \left\{ x = \frac{p}{q} : (p, q) \in \mathbb{Z} \times \mathbb{Z}^* \right\}. \quad (1)$$

When it comes to the set of irrational numbers, several approaches are used. A classic definition relies on set operations, $\text{ir}\mathbb{Q} \equiv \mathbb{R} \setminus \mathbb{Q}$, where $\text{ir}\mathbb{Q}$ represents the difference

2. For further details, see [Keller et al. \(2024\)](#), [Kristiansen \(2021\)](#), [Georgiev et al. \(2021\)](#), [Dubickas \(2017\)](#), [Wu \(2008\)](#), [Chuan \(2005\)](#), [Rudykh and Shulga \(2024\)](#) and [Tong \(1990\)](#), among others.

3. To be more precise, the constructive approach in the representation of $\text{ir}\mathbb{Q}$ involves building up step by step from the set of integers (see [Figure 1](#)).

between the set of real numbers and the set of rational numbers. Another approach, more literal, states that $\text{ir}\mathbb{Q} \equiv \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$. These definitions emphasize that irrational numbers are all real numbers that cannot be expressed as an irreducible fraction of two integers⁴. However, this literal approach is sometimes less intuitive.

In this note, we introduce a formal characterization of irrational numbers using a rational-irrational *perturbation* (RIP) decomposition. Our approach has at least two advantages. First, it explicitly reveals the structural relationship between rational and irrational numbers while offering a constructive view of $\text{ir}\mathbb{Q}$. Second, as illustrated in [Figure 1](#), it adopts a pedagogical perspective that encourages a progressive understanding of irrational numbers starting from the familiar concepts of $\mathbb{Z}$, and $\mathbb{Q}$.

Now, we are ready to characterize the set of irrational numbers formally.

Let Ξ be the set of all Arabic digits, $\Xi = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $a_0, a_1, a_2, a_3, \dots \in \Xi$. Let $\epsilon = a_0.a_1a_2a_3\dots$ such that $\forall m \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, n > 1 : a_m a_{m+1} \dots \neq a_{m+n} a_{m+n+1} \dots$. In other words, the term ϵ is an infinite and non-periodic decimal expansion, noted as $\epsilon \sim \text{n.p.}$. Then, we can carefully define the set of irrational numbers as:

$$\text{ir}\mathbb{Q} := \left\{ x' = \frac{p}{q} + \epsilon : (p, q) \in \mathbb{Z} \times \mathbb{Z}^*, \epsilon \sim \text{n.p.} \right\}. \quad (2)$$

Thus, it is immediate that when the irrational perturbation is zero, $\epsilon = 0$, we recover the rationals. This decomposition highlights the central role of rational numbers as an approximative base for understanding irrationals. Furthermore, following the RIP decomposition, it is evident that every irrational number can be viewed as the sum of a rational number p/q and a term ϵ , representing the gap between the rational approximation p/q and the irrational number x' . This gap ensures that x' remains outside $\mathbb{Q}$.

3 Some Numerical Examples

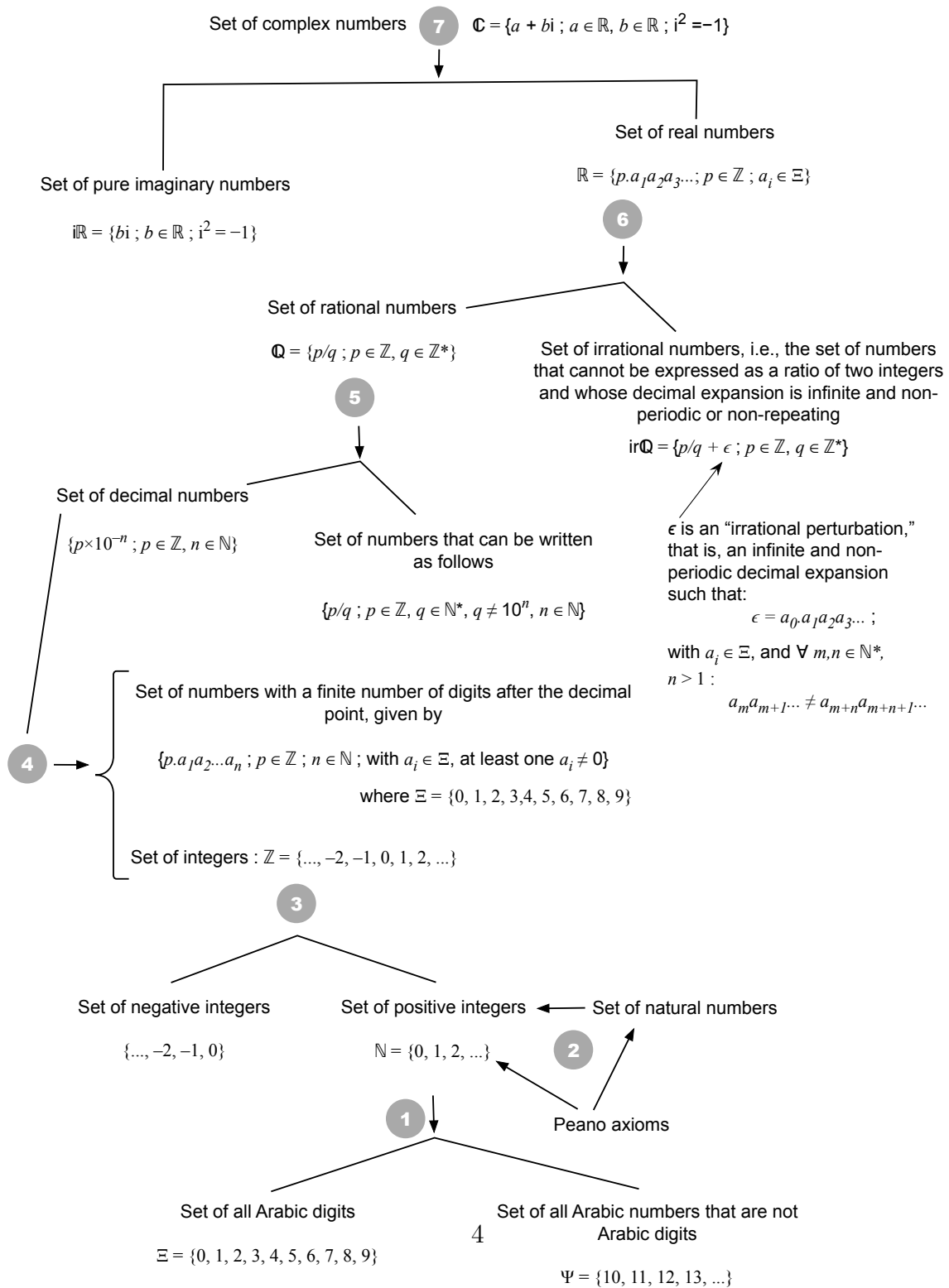
To illustrate the proposed characterization, let us consider three emblematic examples of irrational numbers.

Example 1 (Number $\sqrt{2}$). The square roots of 2 is the positive solution of the equation $x^2 - 2 = 0$ and is approximately equal to 1.4142135623... It is an irrational number because it cannot be expressed as a rational fraction p/q .

Example 2 (Number π). This number represents the ratio of a circle's circumference to its diameter. Its value is approximately 3.14159265359..., with an infinite and

4. We also know that in the interval of $[0, 1]$, the Lebesgue measure of irrational numbers is 1 whereas it is 0 for the rational numbers ([Lai and Danca, 2008](#), p. 246).

Figure 1 – The Classical Numerical Sets



non-periodic expansion.

Example 3 (Number e). The number e , often referred to as Euler's number, is the base of the natural logarithm. It is defined as:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828182846 \dots$$

It is also an irrational number.

Example 4 (Golden Ratio φ). The golden ratio φ is defined as:

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.61803398875 \dots$$

It satisfies the equation $\varphi^2 = \varphi + 1$. φ appears frequently in art, architecture, and nature due to its unique mathematical properties. Its decimal expansion is infinite and non-repeating, confirming its irrationality.

Table 1 – Some Numerical Examples for Illustrating the Consistency of the Characterization in [Equation \(2\)](#).

$x \in \text{ir}\mathbb{Q}$	p	q	a_0	a_1	a_2	a_3	$\dots$	For $m = 1$ and $n = 2^*$: $a_1 a_2 a_3 a_4 \dots \neq a_2 a_3 a_4 a_5 \dots$
$\sqrt{2}$	1	1	0	4	1	4	$\dots$	$4142135623 \dots \neq 1421356237 \dots$
π	3	1	0	1	4	1	$\dots$	$1415926535 \dots \neq 4159265359 \dots$
e	2	1	0	7	1	8	$\dots$	$7182818284 \dots \neq 1828182846 \dots$
φ	1	1	0	6	1	8	$\dots$	$6180339887 \dots \neq 1803398875 \dots$

*Our inequality holds for all $m, n \in \mathbb{N}^*, n > 1$. These numerical examples consider a specific case where $m = 1$ and $n = 2$.

Table 1 offers a detailed illustration of the characterization of irrational numbers developed in this study, using well-known examples such as $\sqrt{2}$, π , e , and the golden ratio φ . These numbers were selected to represent diverse categories of irrational numbers, ranging from solutions to algebraic equations, as seen with $\sqrt{2}$ and φ , to transcendental constants of fundamental importance in mathematics and science, such as π and e .

The table presents the decimal expansion of each number, highlighting their non-periodic nature. The digits are displayed in sequence as $a_1, a_2, a_3, \dots$, and the comparison between consecutive sequences—specifically, $a_1a_2a_3a_4\dots$ and $a_2a_3a_4a_5\dots$ —demonstrates the absence of repeating patterns. This inequality underscores the irrationality of these numbers, consistent with the mathematical characterization developed in the study. Furthermore, the parameters p and q , associated with each number, serve as auxiliary values that allow RIP decomposition and analysis of their decimal structure. The choice of specific sequences for $m = 1$ and $n = 2$ illustrates a particular case of a more general property valid for all $m, n \in \mathbb{N}^*$ where $n > 1$.

This table not only validates the theoretical framework but also provides a concrete and accessible way to observe the distinct properties of irrational numbers. By combining classical examples and a clear focus on their decimal structure, it bridges the gap between abstract theory and tangible mathematical phenomena, enhancing the reader’s understanding of these numbers⁵.

Example 5. The following number $3.312275275275275\dots$ is not irrational. This can be immediately verified by applying the repeating decimal (RIP) decomposition method.

The application of the RIP approach yields the table 2. Then, using a counterexample-based argument, we can refute the statement that $3.314275275275275\dots$ is an irrational number. This can be demonstrated by setting $m = 4$ and $n = 3$, which implies that $a_4a_5a_6a_7\dots = a_7a_8a_9a_{10}\dots$. This equality indicates that the number exhibits a repeating decimal pattern and, therefore, can be expressed as a ratio of two integers, confirming that it is rational.

4 Conclusion

The paper introduces a formal approach for representing $\text{ir}\mathbb{Q}$, using a rational-*irrational perturbation* decomposition. Our step-by-step approach aligns with ped-

5. The analysis presented in this study highlights three key aspects of irrational numbers. First, the non-periodic structure of their decimal expansions is clearly demonstrated by comparing successive sequences, which reveal the absence of repeating patterns. Second, the theoretical consistency of the proposed characterization is validated by the inequality $a_1a_2a_3a_4\dots \neq a_2a_3a_4a_5\dots$, illustrated for $m = 1$ and $n = 2$ as a particular case of a general property valid for all $m, n \in \mathbb{N}^*$ where $n > 1$. Finally, the diversity of examples provided—including $\sqrt{2}$ as a solution to a quadratic equation, π and e as fundamental constants, and φ as the golden ratio—offers a comprehensive exploration of the multifaceted nature of irrational numbers. These findings not only deepen our understanding of these intriguing numbers but also underscore their central role in mathematics and science.

Table 2 – A Supplementary Numerical Examples for Illustrating the Consistency of the Characterization in Equation (2).

$x \in \text{ir}\mathbb{Q}$	p	q	a_0	a_1	a_2	a_3	$\dots$	Counterexample*: for $m = 4$ and $n = 3$, we find that $a_4a_5a_6a_7\dots = a_7a_8a_9a_{10}\dots$
3.3122752752...	3	1	0	3	1	2	...	$2752752\dots = 2752752\dots$

*To prove that a statement of the form $\forall x \in E : P(x)$ is false, we need to find an element x , i.e. an example in the domain of discourse E , for which $P(x)$ is false. Such an example is called a counterexample (Grami 2022, ch. 4).

agogical principles, gradually introducing irrational numbers by building on their connection to the foundational sets $\mathbb{N}$ and $\mathbb{Z}$. The insights from this note could lay the groundwork for further research on numerical sets, with a focus on their didactic value in mathematics education.

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